On kaonic hydrogen. Phenomenological quantum field theoretic model revisited

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Abstract. We argue that due to isospin and U-spin invariance of strong low-energy interactions the S-wave scattering lengths a_0^0 and a_0^1 of $\bar{K}N$ scattering with isospin $I = 0$ and $I = 1$ satisfy the low-energy theorem $a_0^0 + 3a_0^1 = 0$ valid to leading order in chiral expansion. In the model of strong low-energy $\bar{K}N$ interactions at threshold (Eur. Phys. J. A **21**, 11 (2004)) we revisit the contribution of the $\Sigma(1750)$ resonance, which does not saturate the low-energy theorem $a_0^0 + 3a_0^1 = 0$, and replace it by the baryon background with properties of an $SU(3)$ octet. We calculate the S-wave scattering amplitudes of K^-N and K^-d scattering at threshold. We calculate the energy level displacements of the ground states of kaonic hydrogen and deuterium. The result obtained for kaonic hydrogen agrees well with recent experimental data by the DEAR Collaboration. We analyse the cross-sections for elastic and inelastic K^-p scattering for laboratory momenta $70\,\mathrm{MeV}/c < p_K < 150\,\mathrm{MeV}/c$ of the incident K^--meson. The theoretical results agree with the available experimental data within two standard deviations.

PACS. 11.10.Ef Lagrangian and Hamiltonian approach – 11.55.Ds Exact S matrices – 13.75.Gx Pion-baryon interactions – 36.10.-k Exotic atoms and molecules (containing mesons, muons, and other unusual particles)

1 Introduction

Recently in ref. [1] we have proposed a phenomenological quantum field theoretic model for strong low-energy K^-p interactions at threshold for the analysis of the experimental data by the DEAR Collaboration [2,3] on the energy level displacement of the ground state of kaonic hydrogen,

$$-\epsilon_{1s}^{(\mathrm{exp})} + i\,\frac{\Gamma_{1s}^{(\mathrm{exp})}}{2} = (-194 \pm 37\,(\mathrm{stat.}) \pm 6\,(\mathrm{syst.}))$$
$$+ i\,(125 \pm 56\,(\mathrm{stat.}) \pm 15\,(\mathrm{syst.}))\,\mathrm{eV}. \qquad (1.1)$$

According to the Deser-Goldberger-Baumann-Thirring-Trueman formula (the DGBTT), [4], the energy level displacement of the ground state of kaonic hydrogen is related to the S-wave amplitude $f_0^{K^-p}(0)$ of K^-p scattering at threshold as

$$-\epsilon_{1s} + i\,\frac{\Gamma_{1s}}{2} = 2\,\alpha^3\mu^2\,f_0^{K^-p}(0) = 412.13\,f_0^{K^-p}(0), \ (1.2)$$

where $\mu = m_K m_p/(m_K + m_p) = 323.48\,\mathrm{MeV}$ is the reduced mass of the K^-p pair, calculated for $m_K = 493.68\,\mathrm{MeV}$ and $m_p = 938.27\,\mathrm{MeV}$, and $\alpha = 1/137.036$ is the fine-structure constant [5]. The theoretical accuracy of the DGBTT formula eq. (1.2) is about 3% including the vacuum polarization correction [6].

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For a non-zero relative momentum Q the amplitude $f_0^{K^-p}(Q)$ is defined by

$$f_0^{K^-p}(Q) = \frac{1}{2iQ}\left(\eta_0^{K^-p}(Q)\,e^{2i\delta_0^{K^-p}(Q)} - 1\right), \qquad (1.3)$$

where $\eta_0^{K^-p}(Q)$ and $\delta_0^{K^-p}(Q)$ are the inelasticity and the phase shift of the reaction $K^- + p \to K^- + p$, respectively.

The real part $\mathcal{R}e\,f_0^{K^-p}(0)$ of $f_0^{K^-p}(0)$ defines the S-wave scattering length $a_0^{K^-p}$ of K^-p scattering,

$$\mathcal{R}e\,f_0^{K^-p}(0) = a_0^{K^-p} = \frac{1}{2}\,(a_0^0 + a_0^1), \qquad (1.4)$$

where a_0^0 and a_0^1 are the S-wave scattering lengths a_0^I with isospin $I = 0$ and $I = 1$, respectively.

The imaginary part $\mathcal{I}m\,f_0^{K^-p}(0)$ of $f_0^{K^-p}(0)$ is caused by inelastic channels $K^-p \to Y\pi$, where $Y\pi = \Sigma^-\pi^+$, $\Sigma^+\pi^-$, $\Sigma^0\pi^0$ and $\Lambda^0\pi^0$, allowed kinematically at threshold $Q = 0$.

The S-wave amplitude eq. (1.3) can be represented in the following form:

$$\begin{aligned}
f_0^{K^-p}(Q) &= \frac{1}{2iQ}\left(\eta_0^{K^-p}(Q)\,e^{2i\delta_0^{K^-p}(Q)} - 1\right) = \\
&= \frac{1}{2iQ}\left(e^{2i\delta_B^{K^-p}(Q)} - 1\right) \\
&\quad + e^{2i\delta_B^{K^-p}(Q)} f_0^{K^-p}(Q)_R, \qquad (1.5)
\end{aligned}$$

where $\delta_B^{K^-p}(Q)$ is the phase shift of an elastic background of low-energy K^-p scattering and $f_0^{K^-p}(Q)_R$ is the contribution of resonances.

In our model of strong low-energy $\bar{K}N$ interactions near threshold proposed in ref. [1] the imaginary part $\mathcal{I}m\,f_0^{K^0}(0)$ of the S-wave amplitude of K^-p scattering is defined by the contributions of strange baryon resonances $\Lambda(1405)$, $\Lambda(1800)$ and $\Sigma(1750)$. This implies that

$$\mathcal{I}m\,f_0^{K^-p}(0) = \mathcal{I}m\,f_0^{K^-p}(0)_R. \qquad (1.6)$$

According to Gell-Mann's $SU(3)$ classification of hadrons, the $\Lambda(1405)$ resonance is an $SU(3)$ singlet, whereas the resonances $\Lambda(1800)$ and $\Sigma(1750)$ are components of an $SU(3)$ octet [5].

The real part $\mathcal{R}e\,f_0^{K^-p}(0)$ of the S-wave amplitude of K^-p scattering at threshold

$$\mathcal{R}e\,f_0^{K^-p}(0) = \mathcal{R}e\,f_0^{K^-p}(0)_R + \mathcal{R}e\,\tilde{f}_0^{K^-p}(0) \qquad (1.7)$$

is defined by the contribution of i) the strange baryon resonances $\mathcal{R}e\,f_0^{K^-p}(0)_R$ in the s-channel of low-energy elastic K^-p scattering, ii) the exotic four-quark (or $K\bar{K}$ molecules) scalar states $a_0(980)$ and $f_0(980)$ in the t-channel of low-energy elastic K^-p scattering and iii) hadrons with non-exotic quark structures, *i.e.* $q\bar{q}$ for mesons and qqq for baryons, where $q = u, d$ or s

quarks. The contributions of exotic mesons and non-exotic hadrons we denote as $\mathcal{R}e\,\tilde{f}_0^{K^-p}(0)$.

According to ref. [7], we describe strange baryon resonances as elementary particle fields coupled to octets of low-lying baryons $B = (N, \Lambda^0, \Sigma, \Xi)$ and pseudoscalar mesons $P = (\pi, K, \bar{K}, \eta(550))$. The effective phenomenological low-energy Lagrangians of these interactions are [1]:

$$\begin{aligned}
\mathcal{L}_{\Lambda_1 BP}(x) &= g_1\,\bar{\Lambda}_1^0(x)\,\mathrm{tr}\{B(x)P(x)\} + \text{H.c.}, \\
\mathcal{L}_{B_2 BP}(x) &= \frac{1}{\sqrt{2}}\,g_2\,\mathrm{tr}\{\{\bar{B}_2(x), B(x)\}P(x)\} \\
&\quad + \frac{1}{\sqrt{2}}f_2\mathrm{tr}\{[\bar{B}_2(x), B(x)]P(x)\} + \text{H.c.,} \quad (1.8)
\end{aligned}$$

where g_1, g_2 and f_2 are phenomenological coupling constants, $\Lambda_1^0(x)$ and $B_2(x)$ are interpolating field operators of the singlet $\Lambda(1405)$ and octet of strange baryon resonances, respectively. The interactions of resonances with the meson-baryon pairs $\bar{K}N$, $Y\pi$ and $Y\eta(550)$, where $Y = \Sigma^\pm$, Σ^0 or Λ^0, are given by

$$\begin{aligned}
\mathcal{L}_{\Lambda_1^0 BP}(x) &= g_1\,\bar{\Lambda}_1^0(x)(\vec{\Sigma}(x)\cdot\vec{\pi}(x) - p(x)K^-(x) \\
&\quad + n(x)\bar{K}^0(x) + \frac{1}{3}\,\Lambda^0(x)\eta(x)) + \text{H.c.}, \\
\mathcal{L}_{\Lambda_2^0 BP}(x) &= \frac{g_2}{\sqrt{3}}\,\bar{\Lambda}_2^0(x)(\vec{\Sigma}(x)\cdot\vec{\pi}(x) - \Lambda^0(x)\eta(x)) \\
&\quad + \frac{g_2 + 3f_2}{2\sqrt{3}}\,\bar{\Lambda}_2^0(x)\,(p(x)K^-(x) \\
&\quad - n(x)\bar{K}^0(x)) + \text{H.c.}, \\
\mathcal{L}_{\Sigma_2^0 BP}(x) &= f_2\,\bar{\Sigma}_2^0(x)\,(\Sigma^-(x)\pi^+(x) - \Sigma^+(x)\pi^-(x)) \\
&\quad + \frac{g_2}{\sqrt{3}}\,\bar{\Sigma}_2^0(x)\,(\Lambda^0(x)\pi^0(x) + \Sigma^0(x)\eta(x)) \\
&\quad + \frac{g_2 - f_2}{2}\,\bar{\Sigma}_2^0(x)\,(-p(x)K^-(x) \\
&\quad - n(x)\bar{K}^0(x)) + \text{H.c.}, \\
\mathcal{L}_{\Sigma_2^- BP}(x) &= f_2\,\bar{\Sigma}_2^-(x)(\Sigma^-(x)\pi^0(x) - \Sigma^0(x)\pi^-(x)) \\
&\quad + \frac{g_2}{\sqrt{3}}\,\bar{\Sigma}_2^-(x)\Lambda^0(x)\pi^-(x) - \frac{1}{\sqrt{2}}\,(g_2 - f_2) \\
&\quad \times \bar{\Sigma}_2^-(x)n(x)K^-(x) + \text{H.c.} \qquad (1.9)
\end{aligned}$$

As has been pointed out in ref. [7], the inclusion of the $\Lambda(1405)$ resonance as an elementary particle field does not contradict ChPT by Gasser and Leutwyler [8] and allows to calculate the low-energy parameters of $\bar{K}N$ scattering to leading order in Effective Chiral Lagrangians.

Using the effective Lagrangians eq. (1.9) we obtain the S-wave amplitudes of inelastic channels of K^-p scattering at threshold $f(K^-p \to Y\pi)$, where $Y\pi = \Sigma^\mp\pi^\pm$, $\Sigma^0\pi^0$ and $\Lambda^0\pi^0$. The theoretical cross-sections $\sigma(K^-p \to Y\pi)$ for these reactions satisfy the experimental data [9]

see eqs. (1.10) *on the next page*

with an accuracy of about 6% and the constraint that the $\Lambda(1800)$ resonance decouples from the K^-p pair that gives

$$\gamma = \frac{\sigma(K^-p \to \Sigma^-\pi^+)}{\sigma(K^-p \to \Sigma^+\pi^-)} = 2.360 \pm 0.040,$$

$$R_c = \frac{\sigma(K^-p \to \Sigma^-\pi^+) + \sigma(K^-p \to \Sigma^+\pi^-)}{\sigma(K^-p \to \Sigma^-\pi^+) + \sigma(K^-p \to \Sigma^+\pi^-) + \sigma(K^-p \to \Sigma^0\pi^0) + \sigma(K^-p \to \Lambda^0\pi^0)} = 0.664 \pm 0.011,$$

$$R_n = \frac{\sigma(K^-p \to \Lambda^0\pi^0)}{\sigma(K^-p \to \Sigma^0\pi^0) + \sigma(K^-p \to \Lambda^0\pi^0)} = 0.189 \pm 0.015 \qquad (1.10)$$

$f_2 = -g_2/3$. This result is obtained without the specification of the numerical values of the coupling constants g_1 and g_2 and the masses of resonances, but using only physical masses of interacting particles for the calculation of phase volumes. For $f_2 = -g_2/3$ the S-wave amplitudes $f(K^-p \to Y\pi)$ can be defined by

$$f(K^-p \to \Sigma^-\pi^+) = +\frac{1}{4\pi}\frac{\mu}{m_{K^-}}\sqrt{\frac{m_{\Sigma^-}}{m_p}}\left(-A + \frac{1}{2}B\right),$$

$$f(K^-p \to \Sigma^+\pi^-) = +\frac{1}{4\pi}\frac{\mu}{m_{K^-}}\sqrt{\frac{m_{\Sigma^+}}{m_p}}\left(-A - \frac{1}{2}B\right),$$

$$f(K^-p \to \Sigma^0\pi^0) = -\frac{1}{4\pi}\frac{\mu}{m_{K^-}}\sqrt{\frac{m_{\Sigma^0}}{m_p}}\,A,$$

$$f(K^-p \to \Lambda^0\pi^0) = -\frac{1}{4\pi}\frac{\mu}{m_{K^-}}\sqrt{\frac{m_{\Lambda^0}}{m_p}}\frac{\sqrt{3}}{2}B, \qquad (1.11)$$

where $A = -6.02$ fm is the contribution of the $\Lambda(1405)$ resonance, calculated for $g_1 = 0.91$ and $m_{\Lambda(1405)} = 1405$ MeV [1].

The parameter B describes the contribution of the baryon resonance octet. Unfortunately, in ref. [1] we have exaggerated the role of the $\Sigma(1750)$ resonance in strong low-energy $\bar{K}N$ interactions at threshold. The contribution of the $\Sigma(1750)$ resonance with the recommended values of its parameters does not define B correctly. More definitely the contribution of the $\Sigma(1750)$ resonance does not saturate the sum rule eq. (2.13), which is the consequence of the low-energy theorem $a_0^0 + 3\,a_0^1 = 0$ eq. (2.8).

Therefore, instead of the assertion that B is caused by the contribution of the $\Sigma(1750)$ resonance we argue that *B is defined by a contribution of a baryon background with a property of an SU(3) octet and quantum numbers of the $\Lambda(1800)$ and $\Sigma(1750)$ resonances.* The former is important for the correct description of the experimental data eq. (1.10).

Using the relation between the S-wave amplitudes of the reactions $K^-p \to \Sigma^-\pi^+$ and $K^-p \to \Sigma^+\pi^-$, imposed by the experimental data eq. (1.10), we obtain the contribution of the baryon background B in terms of γ, A and the phase volumes of the final $\Sigma^-\pi^+$ and $\Sigma^+\pi^-$ states. This gives

$$B = 2\frac{\sqrt{\gamma\,k_{\Sigma^+\pi^-}} - \sqrt{k_{\Sigma^-\pi^+}}}{\sqrt{\gamma\,k_{\Sigma^+\pi^-}} + \sqrt{k_{\Sigma^-\pi^+}}}\,(-A) = 2.68\,\text{fm}, \qquad (1.12)$$

where $k_{\Sigma^+\pi^-} = 181.34$ MeV and $k_{\Sigma^-\pi^+} = 172.73$ MeV are the relative momenta of the $\Sigma^\pm\pi^\mp$ pairs at threshold of K^-p scattering, calculated for physical masses of interacting particles [5]. The phase volumes of the final $\Sigma^-\pi^+$

and $\Sigma^+\pi^-$ states are equal to $k_{\Sigma^-\pi^+}/4\pi(m_K + m_p)$ and $k_{\Sigma^+\pi^-}/4\pi(m_K + m_p)$, respectively.

The paper is organized as follows. In sect. 2 we calculate the S-wave amplitudes of K^-p and K^-n scattering at threshold. We show that the S-wave scattering lengths $a_0^{K^-p}$ and $a_0^{K^-n}$ of K^-p and K^-n scattering satisfy the low-energy theorem $a_0^{K^-p} + a^{K^-n} = (a_0^0 + 3\,a_0^1)/2 = 0$. We show that in the chiral limit due to isospin invariance $a_0^{K^-p} + a^{K^-n} = (a_0^0 + 3\,a_0^1)/2 = -\sqrt{6}\,b_0^0 = 0$, where $b_0^0 = (a_0^{\pi^-p} + a_0^{\pi^-n})/2$ is the isoscalar S-wave scattering length π^-N scattering, vanishing in the chiral limit. The low-energy theorem $a_0^0 + 3\,a_0^1 = 0$ can be also derived using invariance of strong low-energy interactions under U-spin rotations [10]. We calculate the energy level displacement of the ground state of kaonic hydrogen. The theoretical value agrees well with the experimental data by the DEAR Collaboration. Using the results obtained in sect. 2 for the S-wave scattering lengths of K^-N scattering and in ref. [11] we recalculate the S-wave scattering length $a_0^{K^-d}$ of K^-d scattering at threshold. We calculate the energy level displacement of the ground state of kaonic deuterium. All results agree well with those obtained in ref. [11]. In sect. 3 we analyse the cross-sections for elastic and inelastic K^-p scattering for laboratory momenta $70\,\text{MeV}/c \leq 150\,\text{MeV}/c$ of the incident K^--meson. The theoretical cross-sections agree with the available experimental data within two standard deviations. In the conclusion we discuss the obtained results.

2 S-wave amplitude of K^-N scattering at threshold

2.1 S-wave amplitude of K^-p scattering at threshold

As has been shown in ref. [1], the imaginary part of the S-wave amplitude of K^-p scattering at threshold can be represented by

$$\mathcal{I}m\,f_0^{K^-p}(0) = \mathcal{I}m\,f_0^{K^-p}(0)_R =$$
$$= \frac{1}{R_c}\left(1 + \frac{1}{\gamma}\right)|f(K^-p \to \Sigma^-\pi^+)|^2 k_{\Sigma^-\pi^+}$$
$$= (0.35 \pm 0.02)\,\text{fm}, \qquad (2.1)$$

where $f(K^-p \to \Sigma^-\pi^+) = 0.43\,\text{fm}$ and ± 0.02 is an accuracy of about 6% of our description of the experimental

data eq. (1.11). The contribution of the $\Lambda(1405)$ resonance and the baryon background to $\mathcal{R}e\, f_0^{K^- 0}(0)$ is equal to [1]

$$\mathcal{R}e\, f_0^{K^- p}(0)_R = \frac{1}{4\pi}\,\frac{\mu}{m_K}\,(A+B) = (-0.17 \pm 0.01)\,\mathrm{fm}. \tag{2.2}$$

Since the contribution $\mathcal{R}e\, \tilde{f}_0^{K^- p}(0) = (-0.33 \pm 0.04)\,\mathrm{fm}$, calculated in ref. [1], is not changed, the total real part of the S-wave amplitude of $K^- p$ scattering at threshold amounts to

$$\begin{aligned} \mathcal{R}e\, f_0^{K^- p}(0) &= \mathcal{R}e\, f_0^{K^- p}(0)_R + \mathcal{R}e\, \tilde{f}_0^{K^- p}(0) \\ &= (-0.50 \pm 0.05)\,\mathrm{fm}. \end{aligned} \tag{2.3}$$

Hence, for the S-wave amplitude of $K^- p$ scattering at threshold we get

$$f_0^{K^- p}(0) = (-0.50 \pm 0.05) + i\,(0.35 \pm 0.02)\,\mathrm{fm}. \tag{2.4}$$

This agrees well with the result obtained in ref. [1].

2.2 S-wave amplitude of $K^- n$ scattering at threshold

Since the $K^- n$ pair has isospin $I = 1$, in our model the resonant parts of the S-wave amplitudes of elastic and inelastic $K^- n$ scattering at threshold are described by the contribution of the baryon background B. The imaginary part $\mathcal{I}m\, f_0^{K^- n}(0)$ is defined by the inelastic channels $K^- n \to Y\pi$ with $Y\pi = \Sigma^- \pi^0$, $\Sigma^0 \pi^-$ and $\Lambda^0 \pi^-$. Using the results obtained in ref. [11] we get

$$\begin{aligned} \mathcal{R}e\, f_0^{K^- n}(0) &= \mathcal{R}e\, \tilde{f}_0^{K^- n}(0) + \frac{1}{2\pi}\,\frac{\mu}{m_K}\,B = \\ &= (0.22 \pm 0.02) + \frac{1}{2\pi}\,\frac{\mu}{m_K}\,B \\ &= (0.50 \pm 0.02)\,\mathrm{fm}, \end{aligned}$$

$$\mathcal{I}m\, f_0^{K^- n}(0) = \sum_{Y\pi} |f(K^- n \to Y\pi)|^2 k_{Y\pi} = 0.04\,\mathrm{fm},$$

$$f(K^- n \to \Sigma^- \pi^0) = +\frac{1}{4\pi}\,\frac{\mu}{m_K}\,\sqrt{\frac{m_{\Sigma^-}}{m_p}}\,\frac{1}{\sqrt{2}}\,B = +0.11\,\mathrm{fm},$$

$$f(K^- n \to \Sigma^0 \pi^-) = -\frac{1}{4\pi}\,\frac{\mu}{m_K}\,\sqrt{\frac{m_{\Sigma^0}}{m_p}}\,\frac{1}{\sqrt{2}}\,B = -0.11\,\mathrm{fm},$$

$$f(K^- n \to \Lambda^0 \pi^-) = -\frac{1}{4\pi}\,\frac{\mu}{m_K}\,\sqrt{\frac{m_{\Lambda^0}}{m_p}}\,\frac{\sqrt{3}}{2}\,B = -0.13\,\mathrm{fm}, \tag{2.5}$$

where $k_{\Sigma^- \pi^0} = 181.36\,\mathrm{MeV}$, $k_{\Sigma^0 \pi^-} = 183.50\,\mathrm{MeV}$ and $k_{\Lambda^0 \pi^-} = 256.88\,\mathrm{MeV}$ are the relative momenta of the pairs $\Sigma^- \pi^0$, $\Sigma^0 \pi^-$ and $\Lambda^0 \pi^-$ at threshold of $K^- n$ scattering. Since the contribution of the exotic scalar mesons $a_0(980)$ and $f_0(980)$ to the S-wave scattering amplitude of $K^- n$ scattering at threshold vanishes, $\mathcal{R}e\, \tilde{f}_0^{K^- n}(0) = (0.22 \pm 0.02)\,\mathrm{fm}$ is defined by low-energy interactions of non-exotic hadrons only [11].

The S-wave amplitude of $K^- n$ scattering at threshold is equal to

$$f_0^{K^- n}(0) = (+0.50 \pm 0.02) + i\,(0.04 \pm 0.00)\,\mathrm{fm}. \tag{2.6}$$

Equating $f_0^{K^- p}(0) = (\tilde{a}_0^0 + \tilde{a}_0^1)/2$ and $f_0^{K^- n}(0) = \tilde{a}_0^1$, where $\tilde{a}_0^0$ and $\tilde{a}_0^1$ are complex S-wave scattering lengths of $\bar{K}N$ scattering with isospin $I = 0$ and $I = 1$, we get the numerical values of $\tilde{a}_0^0$ and $\tilde{a}_0^1$:

$$\begin{aligned} \tilde{a}_0^0 &= (-1.50 \pm 0.05) + i\,(0.66 \pm 0.04)\,\mathrm{fm}, \\ \tilde{a}_0^1 &= (+0.50 \pm 0.02) + i\,(0.04 \pm 0.00)\,\mathrm{fm}, \end{aligned} \tag{2.7}$$

where $\mathcal{R}e\, \tilde{a}_0^0 = a_0^0 = (-1.50 \pm 0.05)\,\mathrm{fm}$ and $\mathcal{R}e\, \tilde{a}_0^1 = a_0^1 = (+0.50 \pm 0.02)\,\mathrm{fm}$.

The complex S-wave scattering length $\tilde{a}_0^0$ agrees well with the scattering length obtained by Dalitz and Deloff [12],

$$\tilde{a}_0^0 = (-1.54 \pm 0.05) + i\,(0.74 \pm 0.02)\,\mathrm{fm}$$

for the position of the pole on sheet II of the E-plane $E^* - i\,\Gamma/2$ with $E^* = 1404.9\,\mathrm{MeV}$ and $\Gamma = 53.1\,\mathrm{MeV}$ [12]. This corresponds to our choice of the parameters of the $\Lambda(1405)$ resonance.

We apply the complex S-wave scattering lengths eq. (2.7) to the calculation of the energy level displacement of the ground state of kaonic hydrogen. We use the real parts of these scattering lengths a_0^0 and a_0^1 for the calculation of the energy level shift of the ground state of kaonic deuterium.

2.3 Low-energy theorem $a_0^0 + 3a_0^1 = 0$

The numerical values of the real parts of the S-wave scattering lengths $a_0^{K^- p} = (a_0^0 + a_0^1)/2$ and $a_0^{K^- n} = a_0^1$ of $K^- N$ scattering satisfy the relation

$$a_0^{K^- p} + a_0^{K^- n} = \frac{1}{2}\,(a_0^0 + 3\,a_0^1) = 0. \tag{2.8}$$

This is the low-energy theorem valid in the chiral limit, which can be derived relating the S-wave scattering lengths of $K^- N$ scattering to the S-wave scattering lengths of $\pi^- N$ scattering.

As has been shown by Weinberg [13], in the chiral limit the S-wave scattering lengths $a_0^{\pi^- p} = (2\,a_0^{1/2} + a_0^{3/2})/3$ and $a_0^{\pi^- n} = a_0^{3/2}$ of $\pi^- N$ elastic scattering, where $a_0^{1/2}$ and $a_0^{3/2}$ are the S-wave scattering lengths of πN scattering with isospin $I = 1/2$ and $I = 3/2$, obey the constraint

$$a_0^{\pi^- p} + a_0^{\pi^- n} = \frac{2}{3}\,(a_0^{1/2} + 2a_0^{3/2}) = 2\,b_0^0 = 0, \tag{2.9}$$

which is caused by Adler's consistency condition [14], where b_0^0 is the S-wave scattering length of πN scattering in the t-channel with isospin $I = 0$.

For the derivation of the low-energy theorem eq. (2.8) it is convenient to use the $\mathbb{K}$-matrix approach [15,16]. In terms of the matrix elements of the $\mathbb{K}$-matrix in the t-channel the sum of the S-wave scattering lengths $a_0^{\pi^- p} + a_0^{\pi^- n}$ is equal to

$$\begin{aligned} a_0^{\pi^- p} + a_0^{\pi^- n} &= \langle \pi^+ \pi^- | \mathbb{K} | \bar{p}p + \bar{n}n \rangle \\ &= -\sqrt{\frac{2}{3}}\,\langle I = 0 | \mathbb{K} | I = 0 \rangle, \end{aligned} \tag{2.10}$$

where we have taken into account the isospin properties of the hadronic state $|\bar{p}p + \bar{n}n\rangle$ and $|\pi^-\pi^+\rangle$. Setting

$$\langle I = 0|\mathbb{K}|I = 0\rangle = -\sqrt{6}\, b_0^0 \qquad (2.11)$$

we arrive at the low-energy theorem eq. (2.9).

In terms of the matrix element of the $\mathbb{K}$-matrix in the t-channel the sum of the S-wave scattering lengths $a_0^{K^-p} + a_0^{K^-n}$ can be defined by

$$
\begin{aligned}
a_0^{K^-p} + a_0^{K^-n} &= \langle K^+K^-|\mathbb{K}|\bar{p}p + \bar{n}n\rangle \\
&= \langle I = 0|\mathbb{K}|I = 0\rangle = -\sqrt{6}\, b_0^0 = 0. \quad (2.12)
\end{aligned}
$$

This proves the low-energy theorem eq. (2.8), which is, of course, valid only at leading order in chiral expansion. The former becomes more obvious if one has to derive the low-energy theorem eq. (2.8) using invariance of strong low-energy interactions under U-spin rotations [10]. According to U-spin classification of the components of the pseudoscalar octet [10], the mesons π and K transform as components of doublets (K^+, π^+) and (π^-, K^-). This can be allowed only in the chiral limit.

The relation eq. (2.8) can be rewritten in the form of the sum rule

$$\mathcal{R}e\tilde{f}_0^{K^-p}(0) + \mathcal{R}e\tilde{f}_0^{K^-n}(0) = -\frac{1}{4\pi}\frac{\mu}{m_{K^-}}(A + 3B). \quad (2.13)$$

Using the numerical values $\mathcal{R}e\tilde{f}_0^{K^-p}(0) = -0.33\,\mathrm{fm}$, $\mathcal{R}e\tilde{f}_0^{K^-n}(0) = 0.22\,\mathrm{fm}$, $A = -6.02\,\mathrm{fm}$ and $B = 2.68\,\mathrm{fm}$, one can show that the sum rule eq. (2.13) is fulfilled:

$$
\begin{aligned}
\mathcal{R}e\tilde{f}_0^{K^-p}(0) + \mathcal{R}e\tilde{f}_0^{K^-n}(0) &= -0.11\,\mathrm{fm}\,, \\
-\frac{1}{4\pi}\frac{\mu}{m_{K^-}}(A + 3B) &= -0.11\,\mathrm{fm}.
\end{aligned}
$$

Unfortunately, the $\Sigma(1750)$ resonance does not saturate the sum rule eq. (2.13).

In our model of strong $\bar{K}N$ interactions at threshold the l.h.s. of eq. (2.13) is defined by quark-hadron interactions, whereas the r.h.s. of eq. (2.13) is the resonant part, caused by the contribution of the $\Lambda(1405)$ resonance A and the baryon background B. This is to some extent a manifestation of quark-hadron duality pointed out by Shifman *et al.* within non-perturbative QCD in the form QCD sum rules [17]. Since the l.h.s. of eq. (2.13) can be calculated independently of the assumption of the contribution of the $\Lambda(1405)$ resonance and the baryon background, the sum rule eq. (2.13) places constraints on the parameters of the $\Lambda(1405)$ resonance and the baryon background calculated at leading order in chiral expansion.

2.4 Energy level displacement of the ground state of kaonic hydrogen

For the S-wave amplitude of K^-p scattering at threshold eq. (2.4) the energy level displacement of the ground state

of kaonic hydrogen is equal to

$$
\begin{aligned}
-\epsilon_{1s}^{(0)} + i\frac{\Gamma_{1s}^{(0)}}{2} &= 421.13\, f_0^{K^-p}(0) = 421.13\,\frac{\tilde{a}_0^0 + \tilde{a}_0^1}{2} \\
&= (-205 \pm 21) + i\,(144 \pm 9)\,\mathrm{eV}. \quad (2.14)
\end{aligned}
$$

This result agrees well with the experimental data by the DEAR Collaboration eq. (1.1).

As has been shown in ref. [18], the energy level shift and width of the ground state of kaonic hydrogen acquire the dispersive corrections, caused by the intermediate $\bar{K}^0n$ state on-mass shell

$$
\begin{aligned}
\delta_S^{\mathrm{Disp}} &= \frac{\delta\epsilon_{1s}^{\bar{K}^0n}}{\epsilon_{1s}^{(0)}} = \frac{1}{4}(a_0^1 - a_0^0)^2\, q_0^2 = (8.6 \pm 0.9)\%\,, \\
\delta_W^{\mathrm{Disp}} &= \frac{\delta\Gamma_{1s}^{\bar{K}^0n}}{\Gamma_{1s}^{(0)}} = \frac{1}{2\pi}\frac{(a_0^1 - a_0^0)^2}{\mathcal{I}m\, f_0^{K^-p}(0)\, a_B}\ln\left[\frac{2a_B}{|a_0^0 + a_0^1|}\right] \\
&= (11.1 \pm 1.2)\%\,, \quad (2.15)
\end{aligned}
$$

where $q_0 = \sqrt{2\mu(m_{\bar{K}^0} - m_{K^-} + m_n - m_p)} = 58.35\,\mathrm{MeV}$, calculated for $m_{\bar{K}^0} - m_{K^-} = 3.97\,\mathrm{MeV}$ and $m_n - m_p = 1.29\,\mathrm{MeV}$ [5] and $a_B = 1/\alpha\mu = 83.59\,\mathrm{fm}$ is the Bohr radius.

Taking into account the dispersive corrections eq. (2.15), the energy level displacement of the ground state of kaonic hydrogen is equal to

$$-\epsilon_{1s}^{(\mathrm{th})} + i\frac{\Gamma_{1s}^{(\mathrm{th})}}{2} = (-223 \pm 21) + i\,(159 \pm 9)\,\mathrm{eV}. \quad (2.16)$$

As we have shown above, the S-wave scattering lengths of K^-p and K^-n scattering are calculated at leading order in chiral expansion and satisfy the low-energy theorem eq. (2.8). This allows to take into account contributions caused by next-to-leading order corrections in chiral expansion.

The most important next-to-leading order correction in chiral expansion is the contribution of the $\sigma_{KN}^{(I=1)}(0)$-term, given by [19]

$$
\begin{aligned}
\delta\epsilon_{1s}^{(\sigma)} = \frac{\alpha^3\mu^3}{2\pi m_K F_K^2}&\left[\sigma_{KN}^{(I=1)}(0) - \frac{m_K^2}{4m_N}i\right. \\
&\left.\times \int \mathrm{d}^4x\langle p(\vec{0}, \sigma_p)|\mathrm{T}(J_{50}^{4+i5}(x)J_{50}^{4-i5}(0))|p(\vec{0}, \sigma_p)\rangle\right]. \\
&\quad (2.17)
\end{aligned}
$$

Here $J_{50}^{4\pm i5}(x)$ are time components of the axial-vector hadronic currents $J_{5\mu}^{4\pm i5}(x)$, changing strangeness $|\Delta S| = 1$, $F_K = 113\,\mathrm{MeV}$ is the PCAC constant of the K-meson [5] and the $\sigma_{KN}^{(I=1)}(0)$-term is defined by [20–24]

$$
\begin{aligned}
\sigma_{KN}^{(I=1)}(0) = \frac{m_u + m_s}{4m_N}&\langle p(\vec{0}, \sigma_p)|\bar{u}(0)u(0) \\
&+ \bar{s}(0)s(0)|p(\vec{0}, \sigma_p)\rangle, \quad (2.18)
\end{aligned}
$$

where $u(0)$ and $s(0)$ are operators of the interpolating fields of u and s current quarks [25].

The correction $\delta\epsilon_{1s}^{(\sigma)}$ to the shift of the energy level of the ground state of kaonic hydrogen, caused by the $\sigma_{KN}^{(I=1)}(0)$, is obtained from the S-wave amplitude of K^-p scattering, calculated to next-to-leading order in ChPT expansion at the tree-hadron level [19] and Current Algebra [26,27] (see also ref. [20]):

$$4\pi\left(1+\frac{m_K}{m_N}\right)\tilde{f}_0^{K^-p}(0)=$$
$$\frac{m_K}{F_K^2}-\frac{1}{F_K^2}\sigma_{KN}^{(I=1)}(0)+\frac{m_K^2}{4m_NF_K^2}i\int\mathrm{d}^4x$$
$$\times\langle p(\vec{0},\sigma_p)|\mathrm{T}(J_{50}^{4+i5}(x)J_{50}^{4-i5}(0))|p(\vec{0},\sigma_p)\rangle. \quad (2.19)$$

The contribution of the $\sigma_{KN}^{(I=1)}(0)$-term, $-\sigma_{KN}^{(I=1)}(0)/F_K^2$, to the S-wave amplitude of K^-p scattering in eq. (2.19) has a standard structure [20].

Since the first term m_K/F_K^2, calculated to leading order in chiral expansion, has been already taken into account in ref. [1], the second term, $-\sigma_{KN}^{(I=1)}(0)/F_K^2$, and the third one define next-to-leading order corrections in chiral expansion to the S-wave amplitude of K^-p scattering at threshold.

Taking into account the contribution $\delta\epsilon_{1s}^{(\sigma)}$, the total shift of the energy level of the ground state of kaonic hydrogen is equal to

$$\epsilon_{1s}^{(\mathrm{th})}=223\pm21+\frac{\alpha^3\mu^3}{2\pi m_KF_K^2}\sigma_{KN}^{(I=1)}(0)-\frac{\alpha^3\mu^3m_K}{8\pi F_K^2m_N}i$$
$$\times\int\mathrm{d}^4x\langle p(\vec{0},\sigma_p)|\mathrm{T}(J_{50}^{4+i5}(x)J_{50}^{4-i5}(0))|p(\vec{0},\sigma_p)\rangle.$$
$$(2.20)$$

The theoretical estimates of the value of $\sigma_{KN}^{(I=1)}(0)$, carried out within ChPT with a dimensional regularization of divergent integrals, are converged around the number $\sigma_{KN}^{(I=1)}(0)=(200\pm50)\,\mathrm{MeV}$ [22,23]. Hence, the contribution of $\sigma_{KN}^{(I=1)}(0)$ to the energy level shift amounts to

$$\frac{\alpha^3\mu^3}{2\pi m_KF_K^2}\sigma_{KN}^{(I=1)}(0)=(67\pm17)\,\mathrm{eV}. \quad (2.21)$$

The total shift of the energy level of the ground state of kaonic hydrogen is given by

$$\epsilon_{1s}^{(\mathrm{th})}=(290\pm27)-\frac{\alpha^3\mu^3m_K}{8\pi F_K^2m_N}i$$
$$\times\int\mathrm{d}^4x\langle p(\vec{0},\sigma_p)|\mathrm{T}(J_{50}^{4+i5}(x)J_{50}^{4-i5}(0))|p(\vec{0},\sigma_p)\rangle. \quad (2.22)$$

Hence the theoretical analysis of the second term in eq. (2.22) is required for the correct understanding of the contribution of the $\sigma_{KN}^{I=1}(0)$-term to the energy level shift.

Of course, one can solve the inverse problem. Indeed, calculating the contribution of the term

$$\frac{\alpha^3\mu^3m_K}{8\pi F_K^2m_N}i\int\mathrm{d}^4x\,\langle p(\vec{0},\sigma_p)|\mathrm{T}(J_{50}^{4+i5}(x)J_{50}^{4-i5}(0))|p(\vec{0},\sigma_p)\rangle$$
$$(2.23)$$

in eq. (2.20) and using the experimental data on the energy level shift, measured by the DEAR Collaboration eq. (1.1), one can extract the value of the $\sigma_{KN}^{(I=1)}(0)$-term.

2.5 Energy level displacement of the ground state of kaonic deuterium

Using the real parts of the S-wave scattering lengths of K^-N scattering eq. (2.7) we recalculate the S-wave scattering length $a_0^{K^-d}$ of K^-d scattering. As has been shown in ref. [11], the S-wave scattering length $a_0^{K^-d}$ is equal to

$$a_0^{K^-d}=(a_0^{K^-d})_{\mathrm{EW}}+\mathcal{R}e\,\tilde{f}_0^{K^-d}(0), \quad (2.24)$$

where $(a_0^{K^-d})_{\mathrm{EW}}$ is the Ericson-Weise scattering length of K^-d scattering in the S-wave state [11],

$$(a_0^{K^-d})_{\mathrm{EW}}=\frac{1+m_K/m_N}{1+m_K/m_d}\frac{1}{2}(a_0^0+3a_0^1)$$
$$+\frac{1}{4}\left(1+\frac{m_K}{m_d}\right)^{-1}\left(1+\frac{m_K}{m_N}\right)^2$$
$$\times\left((a_0^1)^2+4a_0^0a_0^1-(a_0^0)^2\right)\left\langle\frac{1}{r_{12}}\right\rangle, \quad (2.25)$$

where r_{12} is a distance between two scatterers n and p [15]. In our approach $\langle 1/r_{12}\rangle$ is defined by [11]

$$\left\langle\frac{1}{r_{12}}\right\rangle=\int\mathrm{d}^3x\,\Psi_d^*(\vec{r})\frac{e^{-m_Kr}}{r}\Psi_d(\vec{r})=0.29\,m_\pi, \quad (2.26)$$

where $\Psi_d(\vec{r})$ is the wave function of the deuteron in the ground state.

We would like to remind that Ericson and Weise did not investigate the K^-d scattering. They analysed only π^-d scattering [15]. However, since the structure of the contribution, given by eq. (2.25), is very similar to that of π^-d scattering, we call such a contribution as the Ericson-Weise scattering length $(a_0^{K^-d})_{\mathrm{EW}}$, which has been derived in ref. [11] at the quantum field theoretic level.

The term $\mathcal{R}e\,\tilde{f}_0^{K^-d}(0)$ in eq. (2.24) is defined by the inelastic two-body and three-body channels of the K^-d scattering at threshold. As has been shown in ref. [11], the contribution of this term is negligible in comparison with the Ericson-Weise scattering length $(a_0^{K^-d})_{\mathrm{EW}}$. Dropping the contribution of this term we get

$$a_0^{K^-d}=(a_0^{K^-d})_{\mathrm{EW}}=-0.47\,\mathrm{fm}. \quad (2.27)$$

Since the imaginary part of the S-wave amplitude of K^-d scattering at threshold is not changed, using the results of ref. [11] we obtain

$$f_0^{K^-d}(0)=(-0.47\pm0.07)+i(0.52\pm0.08)\,\mathrm{fm}. \quad (2.28)$$

This gives the energy level displacement for the ground state of kaonic deuterium equal to ref. [11]:

$$-\epsilon_{1s}+i\frac{\Gamma_{1s}}{2}=601.56\,f_0^{K^-d}(0)$$
$$=(-283\pm42)+i(315\pm48)\,\mathrm{eV}. \quad (2.29)$$

Table 1. Theoretical values of cross-sections for inelastic channels of K^-p scattering. The laboratory momentum of the incident K^--meson is measured in MeV/c and the cross-sections in mb.

p_{lab}	70	80	90	100	110	120	130	140	150	160	170
$\sigma_{\Sigma^-\pi^+}$	83.1	72.3	64.1	57.8	52.8	48.7	45.3	42.5	40.1	38.1	36.3
$\sigma_{\Sigma^+\pi^-}$	36.9	32.1	28.4	25.6	23.4	21.6	20.0	18.8	17.7	16.8	16.0
$\sigma_{\Sigma^0\pi^0}$	60.9	52.9	46.9	42.2	38.5	35.5	33.0	31.0	29.2	27.7	26.4
$\sigma_{\Lambda^0\pi^0}$	12.8	11.1	9.8	8.8	8.0	7.3	6.8	6.3	5.9	5.6	5.3

The value of the S-wave amplitude of K^-d scattering at threshold as well as of the energy level displacement of the ground state of kaonic deuterium agree well with the results obtained in ref. [11].

3 Cross-sections for low-energy K^-p scattering

In this section we apply our model of strong K^-N interactions at threshold to the description of the experimental data on the cross-sections for the reactions $K^-p \to K^-p$ and $K^-p \to Y\pi$, where $Y\pi = \Sigma^\mp\pi^\pm$, $\Sigma^0\pi^0$ and $\Lambda^0\pi^0$, as functions of a laboratory momentum p_{lab} of the incident K^--meson. The available experimental data of the cross-sections are given for the laboratory momenta $50\,\text{MeV}/c \le p_{\text{lab}} \le 250\,\text{MeV}/c$ [28]. This corresponds to relative momenta $40\,\text{MeV}/c \le k \le 200\,\text{MeV}/c$ of the K^-p pair.

We analyse the cross-sections for the reactions $K^-p \to K^-p$ and $K^-p \to Y\pi$ only for the laboratory momenta $70\,\text{MeV}/c \le p_{\text{lab}} \le 150\,\text{MeV}/c$ of the incident K^-, where the experimental data are most reliable. For these momenta the S-wave amplitudes of the inelastic reactions $K^-p \to Y\pi$ are described well by the S-wave scattering lengths

$$
\begin{aligned}
f(K^-p \to \Sigma^-\pi^+) &= a_{\Sigma^-\pi^+} = +0.43\,\text{fm}\,, \\
f(K^-p \to \Sigma^+\pi^-) &= a_{\Sigma^+\pi^-} = +0.28\,\text{fm}\,, \\
f(K^-p \to \Sigma^0\pi^0) &= a_{\Sigma^0\pi^0} = +0.36\,\text{fm}\,, \\
f(K^-p \to \Lambda^0\pi^0) &= a_{\Lambda^0\pi^0} = -0.14\,\text{fm}\,.
\end{aligned} \tag{3.1}
$$

For laboratory momenta $70\,\text{MeV}/c \le p_{\text{lab}} \le 150\,\text{MeV}/c$, due to smallness of the S-wave scattering lengths $a_{Y\pi}$, the cross-sections are equal to

$$
\begin{aligned}
\sigma_{\Sigma^-\pi^+}(k) &= 4\pi\,\frac{k_{\Sigma^-\pi^+}(k)}{k}\,C_0^2(k)\,a_{\Sigma^-\pi^+}^2\,, \\
\sigma_{\Sigma^+\pi^-}(k) &= 4\pi\,\frac{k_{\Sigma^+\pi^-}(k)}{k}\,C_0^2(k)\,a_{\Sigma^+\pi^-}^2\,, \\
\sigma_{\Sigma^0\pi^0}(k) &= 4\pi\,\frac{k_{\Sigma^0\pi^0}(k)}{k}\,C_0^2(k)\,a_{\Sigma^0\pi^0}^2\,, \\
\sigma_{\Lambda^0\pi^0}(k) &= 4\pi\,\frac{k_{\Lambda^0\pi^0}(k)}{k}\,C_0^2(k)\,a_{\Lambda^0\pi^0}^2\,,
\end{aligned} \tag{3.2}
$$

where $C_0^2(k)$ is the contribution of the Coulomb interaction of the K^-p pair

$$
C_0^2(k) = \frac{2\pi\alpha\mu}{k}\,\frac{1}{1 - e^{-2\pi\alpha\mu/k}}\,. \tag{3.3}
$$

The cross-sections for inelastic channels agree well with those obtained in refs. [29,30] (see also ref. [31])). The calculation of the cross-sections eq. (3.2), taking into account the Coulomb interaction in the initial and final state, one can carry out within the potential model approach with strong low-energy interactions described by the effective zero-range potential [32]:

$$
V(\vec{r}) = -\frac{2\pi}{\mu}\,a_{Y\pi}\,\delta^{(3)}(\vec{r})\,, \tag{3.4}
$$

where $a_{Y\pi}$ is a S-wave scattering length of the inelastic channel under consideration. The S-wave amplitude $f(\vec{k}, \vec{k}_{Y\pi})$ of the inelastic channel $K^-p \to Y\pi$ is defined by the spatial integral

$$
\begin{aligned}
f(\vec{k}, \vec{k}_{Y\pi}) &= -\frac{\mu}{2\pi}\int \mathrm{d}^3x\,e^{-i\,\vec{k}_{Y\pi}\cdot\vec{r}}\,V(\vec{r})\,\psi_{K^-p}^C(\vec{k}, \vec{r}) = \\
&= a_{Y\pi}\,e^{\pi/2ka_B}\,\Gamma(1 - i/ka_B)\,.
\end{aligned} \tag{3.5}
$$

Here $\psi_{K^-p}^C(\vec{k}, \vec{r})$ is the exact non-relativistic Coulomb wave function of the relative motion of the K^-p pair in the incoming scattering state with a relative momentum $\vec{k}$. It is given by [33]

$$
\begin{aligned}
\psi_{K^-p}^C(\vec{k}, \vec{r}) &= e^{\pi/2ka_B}\,\Gamma(1 - i/ka_B) \\
&\times e^{i\,\vec{k}\cdot\vec{r}}\,F(i/ka_B, 1, ikr - i\,\vec{k}\cdot\vec{r})\,,
\end{aligned} \tag{3.6}
$$

where $F(i/ka_B, 1, ikr - i\,\vec{k}\cdot\vec{r})$ is the confluent hypergeometric function [33,34].

The numerical values of the theoretical cross-sections for the reactions $K^-p \to \Sigma^-\pi^+$, $K^-p \to \Sigma^+\pi^-$, $K^-p \to \Sigma^0\pi^0$ and $K^-p \to \Lambda^0\pi^0$, calculated for the experimental values of the masses of interacting hadrons [5], are adduced in table 1 and the experimental data are given in table 2 [35] and table 3 [36]. The cross-sections as functions of the laboratory momentum of the incident K^--meson are represented in fig. 1. It is seen that theoretical cross-sections agree with experimental data within two standard deviations.

For the S-wave scattering length $a_{\Lambda^0\pi^0} = -0.14\,\text{fm}$ of the inelastic $K^-p \to \Lambda^0\pi^0$ reaction we calculate the S-wave phase shift of $\Lambda\pi$ scattering at threshold of the $\bar{K}N$ pair $\delta_S^{\Lambda^0\pi^0} = a_{\Lambda^0\pi^0}k_{\Lambda^0\pi^0} = -10.3^0$. This agrees well with recent results obtained by Tandean *et al.* [37] (see their fig. 3 and take the value of the phase shift of $\Lambda\pi$ scattering at threshold of the $\bar{K}N$ pair production).

Table 2. Experimental data on the cross-sections for the reactions $K^-p \to \Sigma^-\pi^+$ and $K^-p \to \Sigma^+\pi^-$. The laboratory momentum of the incident K^--meson is measured in MeV/c and the cross-sections in mb.

p_{lab}	90–110	110–130	130–150
$\sigma_{\Sigma^-\pi^+}$	68 ± 8	60 ± 6	46 ± 4
$\sigma_{\Sigma^+\pi^-}$	34 ± 5	23 ± 4	26 ± 3

Table 3. Experimental data on the cross-sections for the reactions $K^-p \to \Sigma^0\pi^0$ and $K^-p \to \Lambda^0\pi^0$. The laboratory momentum of the incident K^--meson is measured in MeV/c and the cross-sections in mb.

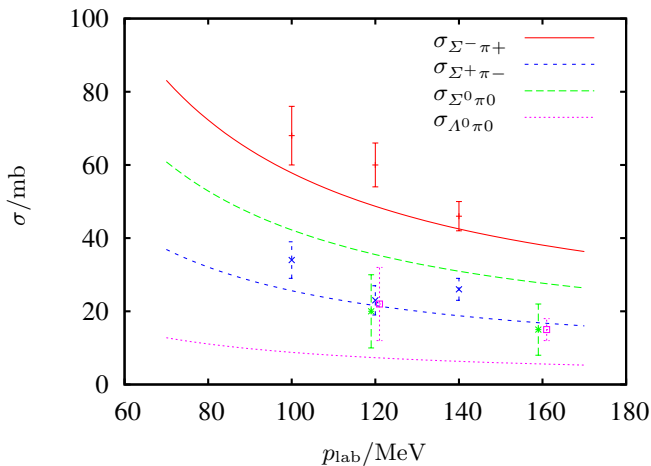| p_{lab}	120	160
$\sigma_{\Sigma^0\pi^0}$	20 ± 10	15 ± 7
$\sigma_{\Lambda^0\pi^0}$	22 ± 10	15 ± 3

Fig. 1. Cross-sections for the inelastic reactions $K^-p \to Y\pi$, where $Y\pi = \Sigma^-\pi^+, \Sigma^+\pi^-, \Sigma^0\pi^0$ and $\Lambda^0\pi^0$.

Due to a contribution of the pure Coulomb interaction to the S-wave amplitude of elastic K^-p scattering only a differential cross-section for elastic K^-p scattering is well defined. For the analysis of experimental data the differential cross-section for elastic K^-p scattering has been taken in the form [31] (see also refs. [29,30])

$$\frac{\mathrm{d}\sigma_{pK^-}^{e\ell}(k)}{\mathrm{d}\Omega} = \left| \frac{\sec^2(\theta/2)}{2k^2 a_B} \exp\left[\frac{2i}{ka_B} \sin(\theta/2) \right] + C_0^2(k) R \exp(i\alpha) \right|^2, \qquad (3.7)$$

where R and α are the experimental fit parameters [31].

For the momenta $100\,\mathrm{MeV}/c \leq p_{\mathrm{lab}} \leq 175\,\mathrm{MeV}/c$ the experimental values of the fit parameters, obtained in ref. [31], are $R = (0.81 \pm 0.06)\,\mathrm{fm}$ and $\alpha = (78 \pm 31)^\circ$. In our model the theoretical values of these parameters are equal to $R = |a_0^{K^-p}| = (0.50 \pm 0.05)\,\mathrm{fm}$ and $\alpha = 180^\circ$. However, the experimental values for the cross-section for elastic K^-p scattering, obtained in ref. [38] for momenta $100\,\mathrm{MeV}/c \leq p_{\mathrm{lab}} \leq 160\,\mathrm{MeV}/c$, are by a factor

1.5 smaller than the data by Humphrey and Ross [31]. This implies that the parameter R can be reduced to the value $R \approx 0.67$, which agrees better with our prediction.

The cross-sections for elastic and inelastic K^-p scattering eq. (3.2), defined for the momenta $70\,\mathrm{MeV}/c \leq p_{\mathrm{lab}} \leq 150\,\mathrm{MeV}/c$, do not contradict the theoretical results obtained by Borasoy *et al.* [28]. The agreement of the theoretical predictions for the cross-sections of elastic and inelastic K^-p scattering is qualitative within about two standard deviations. However, due to self-consistency of our calculation of the S-wave amplitudes of K^-N scattering at threshold and the agreement with the experimental data by the DEAR Collaboration, we can argue that the experimental values of the cross-sections for elastic and inelastic channels of K^-p scattering as well as for K^-n scattering should be remeasured. The same recommendation has been pointed out by Borasoy *et al.* [28].

4 Conclusion

We have revisited our phenomenological quantum field theoretic model of strong low-energy $\bar{K}N$ interactions at threshold. The main change concerns the replacement of the contribution of the $\Sigma(1750)$ resonance with quantum numbers $I\left(J^P\right) = 1\left(\frac{1}{2}^-\right)$ by the baryon background with the same quantum numbers and $SU(3)$ properties. We remind that according to Gell-Mann's classification of hadrons, the $\Sigma(1750)$ resonance belongs to an $SU(3)$ octet of baryons. Following our previous analysis of strong low-energy $\bar{K}N$ interactions [1] and assuming that the S-wave amplitudes of inelastic channels of K^-p scattering at threshold are fully defined by the contribution of the $\Lambda(1405)$ resonance with quantum numbers $I\left(J^P\right) = 0\left(\frac{1}{2}^-\right)$ and the octet of baryon background with $J^P = \frac{1}{2}^-$ we describe the experimental data on ratios of the cross-sections for inelastic channels of K^-p scattering eq. (1.10) within an accuracy of about 6%. Since the non-resonant parts of the S-wave amplitudes are not changed, we have used them and calculated the complex S-wave scattering lengths $\tilde{a}_0^0$ and $\tilde{a}_0^1$ of $\bar{K}N$ scattering with isospin $I = 0$ and $I = 1$, given by eq. (2.7). The complex S-wave scattering length $\tilde{a}_0^0$ agrees well with that obtained by Dalitz and Deloff [12].

It is interesting to notice that the complex S-wave scattering length $\tilde{a}_0^0$, calculated in our model, does not contradict the result obtained by Akaishi and Yamazaki [39] under the assumption that the $\Lambda(1405)$ resonance is the bound K^-p state.

The real parts of the complex S-wave scattering lengths $a_0^{K^-p} = (a_0^0 + a_0^1)/2$ and $a_0^{K^-n} = a_0^1$ of K^-N scattering satisfy the low-energy theorem eq. (2.8). As we have shown above, this low-energy theorem is a $\bar{K}N$ scattering version of the well-known low-energy theorem for the S-wave scattering lengths of π^-N scattering by Weinberg [13].

The low-energy theorem eq. (2.8) can be rewritten in the form of the sum rule eq. (2.13), where the l.h.s. is defined by quark-hadron interactions, whereas the r.h.s.

is the resonant part caused by the contribution of the $\Lambda(1405)$ resonance A and the baryon background B. The sum rule eq. (2.13) can be accepted as a manifestation of quark-hadron duality pointed out by Shifman *et al.* [17] within non-perturbative QCD in the form of QCD sum rules.

Since in our model the S-wave scattering lengths are calculated to leading order in chiral expansion and satisfy the low-energy theorem eq. (2.8), we can argue that our model is self-consistent to leading order in chiral expansion. This implies that the inclusion of the contributions of next-to-leading order corrections in chiral expansion is well defined and allows to provide the investigation of the contribution of the $\sigma_{KN}^{I=1}(0)$-term to the S-wave scattering length of K^-p and $\bar{K}^-d$ scattering and the energy level displacements of the ground states of kaonic atoms.

The analysis of the contribution of the $\sigma_{KN}^{I=1}(0)$-term demands the calculation of the quantity, defined by eq. (2.23),

$$\frac{\alpha^3\mu^3 m_K}{8\pi F_K^2 m_N}\, i \int \mathrm{d}^4x\, \langle p(\vec{0},\sigma_p)|\mathrm{T}(J_{50}^{4+i5}(x)J_{50}^{4-i5}(0))|p(\vec{0},\sigma_p)\rangle.$$

We are planning to carry out this calculation in our forthcoming publication.

The energy level displacement of the ground state of kaonic hydrogen eq. (2.14), calculated for the complex S-wave scattering lengths eq. (2.7), agrees well with the result obtained in ref. [1] and the experimental data by the DEAR Collaboration. The account for the contribution of the dispersive corrections, caused by the intermediate $\bar{K}^0n$ state on-mass shell [18], changes the values of the energy level shift and width by about 8%.

We have recalculated the S-wave scattering length $a_0^{K^-d}$ of K^-d scattering for the new values of the S-wave scattering lengths a_0^0 and a_0^1 obeying the low-energy theorem $a_0^0 + 3\,a_0^1 = 0$. We have shown that the obtained result is not changed with respect to that calculated in ref. [11].

For the confirmation of the self-consistency our approach we have analysed the cross-sections for elastic and inelastic channels of K^-p scattering for laboratory momenta $70\,\mathrm{MeV}/c \leq p_{\mathrm{lab}} \leq 150\,\mathrm{MeV}/c$ of the incident K^--meson. We have shown that the cross-sections for the reactions $K^-p \to K^-p$ and $K^-p \to Y\pi$, which we have calculated by using the S-wave amplitudes of elastic and inelastic channels of K^-p scattering at threshold, do not contradict the experimental data within two standard deviations. However, the constraints imposed by recent experimental data by the DEAR Collaboration demand a revision of these data.

The energy level displacement of the ground state of kaonic hydrogen has been analysed in ref. [28] and ref. [40]. The result predicted by Borasoy *et al.* [28] within the $SU(3)$ chiral effective Lagrangian approach with relativistic coupled channels technique is equal to

$$-\epsilon_{1s} + i\,\frac{\Gamma_{1s}}{2} = 412.13\, f_0^{K^-p}(0) = -235 + i\,195\,\mathrm{eV}, \quad (4.1)$$

where $f_0^{K^-p}(0) = -0.57 + i\,0.47\,\mathrm{fm}$. It has been obtained as *a result of an "optimal" compromise between the various existing data sets* [28]. The energy level displacement of the ground state of kaonic hydrogen, obtained in ref. [28], agrees with the experimental data by the DEAR Collaboration within experimental error bars. Our result for the energy level shift agrees well with that obtained in ref. [28], whereas the agreement for the values of the energy level width is only within an accuracy of about 30%.

The energy level displacement of the ground state of kaonic hydrogen has been calculated by Meißner *et al.* [40] under the assumption of the dominant role of the $\bar{K}^0n$-cusp. Such a hypothesis has been proposed by Dalitz and Tuan in 1960 [29] (see also ref. [30]) in the $\mathbb{K}$-matrix approach in the zero-range approximation. Meißner *et al.* have argued that the S-wave amplitude

$$\tilde{f}_0^{K^-p}(0) = \frac{\dfrac{\tilde{a}_0^0 + \tilde{a}_0^1}{2} + q_0\,\tilde{a}_0^0\tilde{a}_0^1}{1 + \left(\dfrac{\tilde{a}_0^0 + \tilde{a}_0^1}{2}\right)q_0}, \quad (4.2)$$

obtained by Dalitz and Tuan within the $\mathbb{K}$-matrix approach in the zero-range approximation refs. [29,30], can be derived within a non-relativistic effective Lagrangian approach based on ChPT by Gasser and Leutwyler [8].

For the complex S-wave scattering lengths eq. (2.7), the energy level displacement of the ground state of kaonic hydrogen, caused by the $\bar{K}^0n$-cusp, is equal to

$$-\epsilon_{1s} + i\,\frac{\Gamma_{1s}}{2} = 412.13\, f_0^{K^-p}(0) = -325 + i\,248\,\mathrm{eV}, \quad (4.3)$$

where $f_0^{K^-p}(0) = -0.78 + i\,0.60\,\mathrm{fm}$. This result agrees well with the experimental data by the KEK Collaboration [41]

$$-\epsilon_{1s}^{(\mathrm{exp})} + i\,\frac{\Gamma_{1s}^{(\mathrm{exp})}}{2} = (-323\pm64) + i\,(204\pm115)\,\mathrm{eV}. \quad (4.4)$$

Thus, as has been pointed out by Gasser [42]: *... the theory of $\bar{K}p$ scattering leaves many questions open. More precise data will reveal whether present techniques are able to describe the complicated situation properly.*

A new set of measurements by the DEAR/SIDDHARTA Collaborations [3], which is planned for 2006 year and intended for to reach a precision of the experimental data on the energy level displacement of the ground state of kaonic hydrogen and kaonic deuterium at the eV level, should place constraints on theoretical approaches to the description of strong low-energy $\bar{K}N$ interactions at threshold.

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